A Comparison of Three Perturbation Methods for Earth-Moon-Spaceship Problem

Ali Hasen Nayfeh*
Stanford University, Stanford, Calif.

The limiting case of the restricted three-body problem, in which the mass of one of the finite bodies is much smaller than the mass of the other, is of the singular perturbation type. The first-order perturbation solution has a logarithmic singularity at the position of the smaller body, and higher approximations are increasingly more singular. Three methods of treating singular perturbation problems are compared as applied to this problem. Uniformly valid first-order solutions are obtained for the problem of a two-fixed force-center by using the generalized method. It is shown that the generalized method gives better approximations than the method of inner and outer expansions and thus can be used for wider ranges of the small parameter. Furthermore, the Poincare-Lighthill-Kuo (PLK) method is shown to give incorrect results for the trajectory.

1. Introduction

THE limiting case of the restricted three-body problem, in which the mass of one of the finite bodies (the earth) is much larger than the mass of the other body (the moon), is treated by perturbation methods for finite times. However, the straightforward first-order perturbation solution is not uniformly valid because it has a logarithmic singularity at the position of the moon, and higher approximations are increasingly more singular in the region of nonuniformity. Hence, this problem is of the singular perturbation type.

This problem has been treated previously by Lagerstrom and Kevorkian[2,3] and Perko[8] by using the method of matched asymptotic expansions (the method of inner and outer expansions).[1] Lagerstrom and Kevorkian obtained uniformly valid first-order solutions for the problem of a two-fixed force-center[2] and the restricted three-body problem.[3] Subsequently, Perko[8] obtained a uniformly valid first-order solution for the restricted three-body problem using orbital variables by applying the same method.

Here, we investigate two alternative methods of treating singular perturbation problems: the method of straining of coordinates (PLK method)[4,5,9] and the generalized method for treating singular perturbation problems.[7] We will restrict our considerations here to the problem of the two-fixed force-center and compare our results with those of Lagerstrom and Kevorkian.[2]

2. Formulation of Problem

A spaceship of mass m is moving in the gravitational field of two-fixed mass centers. The mass M_e of the earth is much larger than the mass M_m of the moon. With respect to the rectangular Cartesian coordinate system shown in Fig. 1, the differential equations of motion are[6]

$$d^2\bar{\mathbf{r}}/d\bar{t}^2 = G\,\mathrm{grad}[(M_e/\bar{r}_e) + (M_m/\bar{r}_m)] \tag{2.1a}$$

where

$$\bar{r}_e^2 = \bar{x}^2 + \bar{y}^2 \qquad \bar{r}_m^2 = (\bar{x} - d)^2 + \bar{y}^2 \tag{2.1b}$$

d is the distance between the two mass centers, and G is the universal gravitational constant.

The nondimensional equations of motion are

$$d^2x/dt^2 = -(1-\mu)(x/r_e^3) - \mu(x - 1/r_m^3) \tag{2.2a}$$

$$d^2y/dt^2 = -(1-\mu)(y/r_e^3) - \mu(y/r_m^3) \tag{2.2b}$$

where

$$x = \bar{x}/d \qquad y = \bar{y}/d \qquad r = \bar{r}/d \tag{2.3a}$$

$$\mu = M_m/M_e + M_m \ll 1 \tag{2.3b}$$

$$t = \bar{t}[d^3/G(M_m + M_e)]^{-1/2} \tag{2.3c}$$

These equations must be supplemented by initial conditions. The initial conditions are chosen here to be identical to those of Lagerstrom and Kevorkian[2] in order to facilitate the comparison of their results and ours

$$x = 0 \qquad y = 0 \qquad dy/dx = -\mu c \qquad \text{at} \qquad t = 0 \tag{2.4a}$$

$$h = -\rho^2 \qquad \rho \neq 1 \tag{2.4b}$$

where h is the total energy of the spaceship.

3. Methods of Solution

Although the problem considered can be solved by quadratures because of the existence of two integrals, we will use perturbation methods, which can be applied to the restricted three-body problem where only one invariant integral is known.[10] However, the straightforward perturbation expansion is not uniformly valid as shown below. If we interchange the roles of x and t, the initial conditions suggest straightforward expansions of the form

$$t = \sum_{n=0}^{\infty} \mu^n t_n(x) \tag{3.1a}$$

$$y = \sum_{n=1}^{\infty} \mu^n y_n(x) \tag{3.1b}$$

Substituting (3.1) in (2.2) and equating coefficients of equal powers of μ gives

$$t_0''/t_0'^3 = 1/x^2 \tag{3.2a}$$

$$-(t_1''/t_0'^3) + (3t_0''t_1'/t_0'^4) = (1/x^2) + [1/(1-x)^2] \tag{3.2b}$$

$$(y_1''/t_0'^2) - (t_0''y_1'/t_0'^3) = -y_1/x^2 \tag{3.2c}$$

Solving these equations subject to the initial conditions yields

$$2^{1/2}t_0 = (1/\rho^3)\sin^{-1}\rho(x)^{1/2} - (1/\rho^2)[x(1-\rho^2x)]^{1/2} \tag{3.3a}$$

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* Research Assistant, Department of Aeronautics and Astronautics; now Senior Scientist, Heliodyne Corporation, Los Angeles, Calif.

$$2^{1/2}t_1 = -\frac{2}{\rho^3}\sin^{-1}\rho(x)^{1/2} + \frac{2-\rho^2}{\rho^2(1-\rho^2)}\left[\frac{x}{1-\rho^2 x}\right]^{1/2} + \frac{1}{2(1-\rho^2)^{3/2}} \times \ln\frac{1+(1-2\rho^2)x - 2[x(1-\rho^2)(1-\rho^2 x)]^{1/2}}{1-x} \tag{3.3b}$$

$$y_1 = -cx \tag{3.3c}$$

$$t_2 = 0[1/(1-x)] \qquad \text{as} \qquad x \to 1 \tag{3.3d}$$

From (3.3), it is seen that this straightforward expansion is not valid when $(1-x) = 0(\mu)$.

3.1 PLK Method[4,5,9]

We will seek a uniformly valid solution by straining the coordinate x. We let

$$x = s + \mu x_1(s) + \ldots \tag{3.4a}$$

$$t = t_0(s) + \mu t_1(s) + \ldots \tag{3.4b}$$

$$y = \mu y_1(s) + \ldots \tag{3.4c}$$

Thus,

$$\frac{d^2x}{dt^2} = \frac{\mu x_1''}{(t_0' + \mu t_1' + \ldots)^2} - \frac{(1+\mu x_1' + \ldots)(t_0'' + \mu t_1'' + \ldots)}{(t_0' + \mu t_1' + \ldots)^3} \tag{3.5a}$$

$$\frac{d^2y}{dt^2} = \frac{\mu y_1''}{(t_0' + \mu t_1' + \ldots)^2} - \frac{\mu y_1' t_0''}{(t_0' + \mu t_1' + \ldots)^3} \tag{3.5b}$$

where the primes denote differentiation with respect to s. We substitute (3.4) in (2.2), equate coefficients of equal powers of μ, and get equations for $t_0, t_1, \ldots,$ and $y_1, y_2, \ldots,$. We determine the functions $x_n(s)$ such that higher approximations shall be no more singular than the first. The solution will be given in implicit form in terms of the parameter s.

However, there is no need for solving for $t_1(s)$ if we want only a uniformly valid first-order solution. It is sufficient to inspect the equation for $t_1(s)$ in order to determine $x_1(s)$ such that $t_1(s)$ shall be no more singular than $t_0(s)$.

3.2 Generalized Method[7]

First, we introduce a new additional, independent variable η. Since the straightforward perturbation expansion breaks down when $1 - x = 0(\mu)$, and since we are interested in first-order solutions, we let

$$\eta = (1-x)/\mu \tag{3.6}$$

Second, the ordinary differential equations (2.2) are transformed into partial differential equations of the two independent variables x and η. The functions of the independent variable x which appear in the origin equations stay unchanged except for those that reflect the nonuniformity. In our problem $(1-x)$ reflects the nonuniformity, and therefore it is replaced by $\mu\eta$ whenever it appears in the equations.

Third, we assume that there exist uniformly valid asymptotic expansions of $t(x;\mu)$ and $y(x;\mu)$ in the form

$$t(x;\mu) = t_0(x,\eta) + \mu t_1(x,\eta) + \mu^2 t_2(x,\eta) + \ldots \tag{3.7a}$$

$$y(x;\mu) = \mu y_1(x,\eta) + \mu^2 y_2(x,\eta) + \ldots \tag{3.7b}$$

where

$$t_0(x,\eta) \qquad y_1(x,\eta) \qquad \frac{t_n(x,\eta)}{t_0(x,\eta)} \qquad \frac{y_n(x,\eta)}{y_1(x,\eta)} < \infty \tag{3.8}$$

for all x and η where x is the domain of interest.

Fourth, we substitute (3.7) in the transformed partial differential equations of (2.2) and equate coefficients of equal powers of μ. We solve the resulting equations. The solution will contain arbitrary functions of the independent variable x. They will be determined either by imposing the condition (3.8) or equivalently by requiring these solutions to reduce to the straightforward expansions (3.3) far from the moon.

4. One-Dimensional Case

Consider first the one-dimensional problem in which the spacecraft is projected from the earth to the moon with zero total energy. Thus, the problem becomes

$$d^2x/dt^2 = -(1-\mu/x^2) + [\mu/(1-x)^2] \tag{4.1a}$$

$$t(0) = 0 \qquad h = 0 \tag{4.1b}$$

Integrating once and using the fact that the total energy is zero gives

$$\tfrac{1}{2}(dx/dt)^2 = (1-\mu/x) + (\mu/1-x) \tag{4.2a}$$

$$t(0) = 0 \tag{4.2b}$$

The exact solution of this simple example is, respectively,

$$2^{1/2}t = \pm\int_0^x \left[\frac{s(1-s)}{(1-\mu)-(1-2\mu)s}\right]^{1/2} ds \qquad \text{for} \quad \frac{dx}{dt} > 0$$

$$dx/dt < 0 \tag{4.3}$$

4.1 PLK Method

Let

$$t = t_0(s) + \mu t_1(s) + \ldots \tag{4.4a}$$

$$x = s + \mu x_1(s) + \ldots \tag{4.4b}$$

Substituting (4.4) in (4.2a) and equating coefficients of equal powers of μ leads to

$$1/2t_0'^2 = 1/s \tag{4.5a}$$

$$t_1'/t_0'^3 = (x_1'/t_0'^2) + (x_1/s^2) + (1/s) - (1/1-s) \tag{4.5b}$$

The solution of (4.5a) is

$$2^{1/2}t_0 = \tfrac{2}{3}s^{3/2} + c_1 \tag{4.6}$$

where c_1 is an arbitrary constant.

The first-order approximation $t_1(s)$ will be no more singular than $t_0(s)$ if the right-hand side of (4.5b) is annihilated. Thus,

$$(x_1'/t_0'^2) + (x_1/s^2) + (1/s) - (1/1-s) = 0 \tag{4.7a}$$

Hence,

$$x_1 = \frac{1}{2}s^{-1/2}\ln\frac{1+s^{1/2}}{1-s^{1/2}} - \frac{2}{3}s + c_2 s^{-1/2} - 1 \tag{4.7b}$$

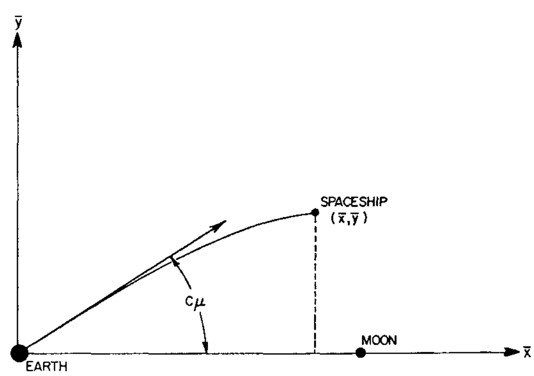

Fig. 1 Coordinate system.

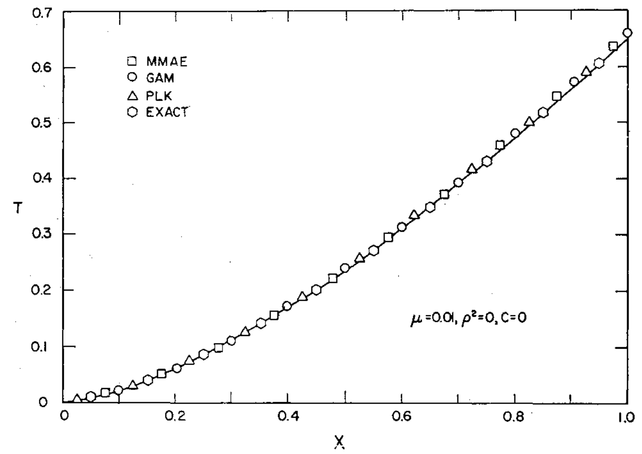

Fig. 2 The time $[T = (2)^{1/2}t]$ taken by the spaceship to travel a distance x in the one-dimensional problem when the total energy is zero, and $\mu = 0.01$.

where c_2 is an arbitrary constant. In order that the straining x_1 be finite when $s = 0$, c_2 must be chosen $= 0$. Using the initial condition $t(0) = 0$ leads to

$$2^{1/2}t = \tfrac{2}{3}s^{3/2} \tag{4.8a}$$

where

$$x = s + \mu\left[\frac{1}{2}\, s^{-1/2}\ln\frac{1+s^{1/2}}{1-s^{1/2}} - \frac{2}{3}\, s - 1\right] \tag{4.8b}$$

4.2 Generalized Method

Since the nonuniformity occurs when $(1 - x) = 0(\mu)$, we introduce an additional independent variable η such that

$$\eta = (1 - x)/\mu \tag{4.9}$$

thereby transforming the ordinary differential equation (4.2a) into

$$2^{1/2}\left(t_x - \frac{1}{\mu}\, t_\eta\right) = \pm\left(\frac{1-\mu}{x} + \frac{1}{\eta}\right)^{-1/2} \tag{4.10}$$

We assume that

$$t = \sum_{n=0}^{\infty} \mu^n t_n(x, \eta) \tag{4.11a}$$

where

$$t_0(x, \eta) \qquad t_n(x, \eta)/t_0(x, \eta) < \infty \tag{4.11b}$$

for

$$0 \leq x \leq 1 \qquad 0 \leq \eta \leq 1/\mu$$

Substituting the series (4.11a) in (4.10) and equating coefficients of equal powers of μ, we obtain

$$t_{0\eta} = 0 \tag{4.12a}$$

$$2^{1/2}(t_{0x} - t_{1\eta}) = (x\eta/\eta + x)^{1/2} \tag{4.12b}$$

$$2^{1/2}(t_{1x} - t_{2\eta}) = \tfrac{1}{2}x^{1/2}(\eta/\eta + x)^{3/2} \tag{4.12c}$$

The solutions of (4.12a) and (4.12b) are

$$2^{1/2}t_0 = A(x) \tag{4.13a}$$

$$2^{1/2}t_1 = A'(x)\eta - x^{1/2}[\eta(\eta + x)]^{1/2} + x^{3/2}\sinh^{-1}(\eta/x)^{1/2} + B(x) \tag{4.13b}$$

where $A(x)$ and $B(x)$ are arbitrary functions of x.

The condition that

$$t_1(x, \eta)/t_0(x, \eta) < \infty \qquad \text{for} \qquad 0 \leq \eta \leq 1/\mu$$

requires that

$$A'(x) = x^{1/2} \tag{4.14a}$$

$$B(x) = \tfrac{1}{2}x^{3/2}\ln\mu + C(x) \tag{4.14b}$$

where $C(x)$ is an arbitrary function of x. From (4.14a), we obtain

$$A(x) = \tfrac{2}{3}x^{3/2} + a \tag{4.15a}$$

where a is an arbitrary constant. From (4.12c), we get

$$2^{1/2}t_{2\eta} = \frac{1}{2}\, x^{-1/2}\,\eta - \frac{1}{2}\, x^{-1/2}[\eta(\eta + x)]^{1/2} - x^{1/2}\left(\frac{\eta}{\eta + x}\right)^{1/2} - \frac{1}{2}\, x^{1/2}\left(\frac{\eta}{\eta + x}\right)^{3/2} + \frac{3}{2}\, x^{1/2}\sinh^{-1}\left(\frac{\eta}{x}\right)^{1/2} + \frac{3}{4}\, x^{1/2}\ln\mu + C'(x) \tag{4.15b}$$

The condition (3.8) necessitates that

$$C'(x) = \tfrac{7}{4}\, x^{1/2} - \tfrac{3}{2}\, x^{1/2}\ln 2 + \tfrac{3}{4}x^{1/2}\ln(x/1 - x) \tag{4.15c}$$

Hence,

$$C(x) = \frac{7}{6}\, x^{3/2} + x^{1/2} + \frac{1}{2}\, x^{3/2}\ln\frac{x}{4(1 - x)} - \frac{1}{2}\ln\frac{1 + x^{1/2}}{1 - x^{1/2}} + c_3 \tag{4.15d}$$

where c_3 is a constant. Imposing the initial condition $t(0) = 0$, we find that

$$2^{1/2}t = \frac{2}{3}\, x^{3/2} + \mu\left\{x^{1/2} + \frac{7}{6}\, x^{3/2} - \frac{1}{2}\ln\frac{1 + x^{1/2}}{1 - x^{1/2}} + x^{1/2}\eta - x^{1/2}[\eta(\eta + x)]^{1/2} + x^{3/2}\ln\frac{\eta^{1/2} + (\eta + x)^{1/2}}{2\eta^{1/2}}\right\} + 0(\mu^2) \tag{4.16}$$

Since $t(x, \eta; \mu)$ is uniformly valid, it must reduce to the straightforward expansion far away from the moon. We can use the previously stated condition to determine the arbitrary functions $A(x)$ and $B(x)$ instead of the condition

$$t_n(x, \eta)/t_0(x, \eta) < \infty \qquad \text{for} \qquad 0 \leq \eta \leq 1/\mu$$

The straightforward perturbation expansion, i.e., $\lim 2^{1/2}t$ as $\mu \to 0$ and x fixed of (4.13a) and (4.13b) are

$$2^{1/2}t_0 = A(x) \tag{4.17a}$$

$$2^{1/2}t_1 = -\frac{1}{2}\, x^{3/2}(1 + \ln x) + \frac{1}{2}\, x^{3/2} \times \left(\ln\frac{1 - x}{\mu} + 2\ln 2\right) + B(x) \tag{4.17b}$$

The straightforward perturbation solution of (4.2a) is

$$2^{1/2}t = \frac{2}{3}\, x^{3/2} + \mu\left(\frac{2}{3}\, x^{3/2} + x^{1/2} - \frac{1}{2}\ln\frac{1 + x^{1/2}}{1 - x^{1/2}}\right) + 0(\mu^2) \tag{4.17c}$$

Comparing (4.17a) and (4.17b) with (4.17c), we obtain

$$A(x) = \tfrac{2}{3}x^{3/2} \tag{4.18a}$$

$$B(x) = \frac{7}{6}\, x^{3/2} + x^{1/2} - \frac{1}{2}\ln\frac{1 + x^{1/2}}{1 - x^{1/2}} + x^{3/2}\ln\frac{1}{2}\left(\frac{x\mu}{1 - x}\right)^{1/2} \tag{4.18b}$$

Hence,

$$2^{1/2}t = \frac{2}{3}x^{3/2} + \mu\left[x^{1/2} + \frac{7}{6}x^{3/2} - \frac{1}{2}\ln\frac{1+x^{1/2}}{1-x^{1/2}} + x^{1/2}\eta - x^{1/2}[\eta(\eta+x)]^{1/2} + x^{3/2}\ln\frac{\eta^{1/2}+(\eta+x)^{1/2}}{2\eta^{1/2}}\right] + 0(\mu)^2 \quad (4.19)$$

which is exactly (4.16). The results of this section are graphed in Figs. 2–4 for comparison with the exact integral and the solution of Lagerstrom and Kevorkian[2]† for μ = 0.25, 0.1, and 0.01, respectively.

5. Two-Dimensional Case

5.1 PLK Method

Substituting (3.4) in (2.2) and equating coefficients of equal powers of μ, we get

$$t_0''/t_0'^3 = 1/s^2 \quad (5.1a)$$

$$\frac{t_1''}{t_0'^3} - \frac{3t_1't_0''}{t_0'^4} = \frac{x_1''}{t_0'^2} - \frac{x_1't_0''}{t_0'^3} - \frac{2x_1}{s^3} - \frac{1}{s^2} - \frac{1}{(1-s)^2} \quad (5.1b)$$

$$(y_1''/t_0'^2) - (y_1't_0''/t_0'^3) = -y_1/s^3 \quad (5.1c)$$

where the prime denotes differentiation with respect to s. The solution of (5.1a), subject to the condition $h = -\rho^2$ is

$$2^{1/2}t_0 = (1/\rho^3)\sin^{-1}\rho s^{1/2} - (1/\rho^2)[s(1-\rho^2 s)]^{1/2} + c_1 \quad (5.2a)$$

where c_1 is an arbitrary constant. The general solution of (5.1c) is

$$y_1 = c_2 s + c_3[s(1-\rho^2 s)]^{1/2} \quad (5.2b)$$

where c_2 and c_3 are arbitrary constants. The first-order approximation $t_1(s)$ will be no more singular than $t_0(s)$ if we annihilate the right-hand terms in (5 1b). Thus,

$$\frac{x_1''}{t_0'^2} - \frac{x_1't_0''}{t_0'^3} - \frac{2x_1}{s^3} - \frac{1}{s^2} - \frac{1}{(1-s)^2} = 0 \quad (5.3a)$$

$$x_1 = \frac{2}{\rho^3}\left(\frac{1-\rho^2 s}{s}\right)^{1/2}\sin^{-1}\rho s^{1/2} - \frac{2-\rho^2}{\rho^2(1-\rho^2)} + \frac{1}{2(1-\rho^2)^{3/2}}\left(\frac{1-\rho^2 s}{s}\right)^{1/2}\ln\frac{(1-\rho^2 s)^{1/2} + [(1-\rho^2)s]^{1/2}}{(1-\rho^2 s)^{1/2} - [(1-\rho^2)s]^{1/2}} \quad (5.3b)$$

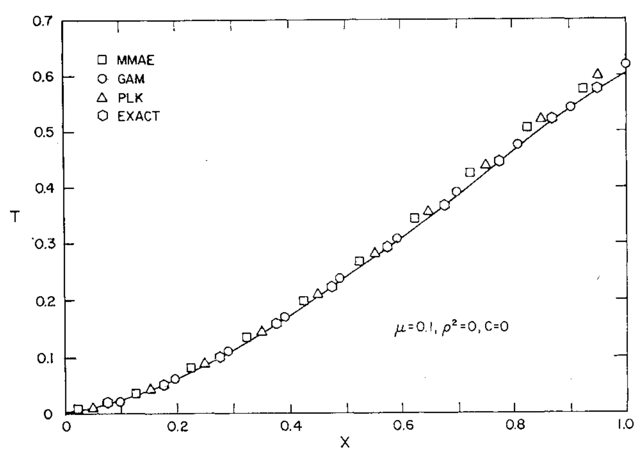

Fig. 3 The time [$T = (2)^{1/2}t$] taken by the spaceship to travel a distance x in the one-dimensional problem when the total energy is zero, and $\mu = 0.1$.

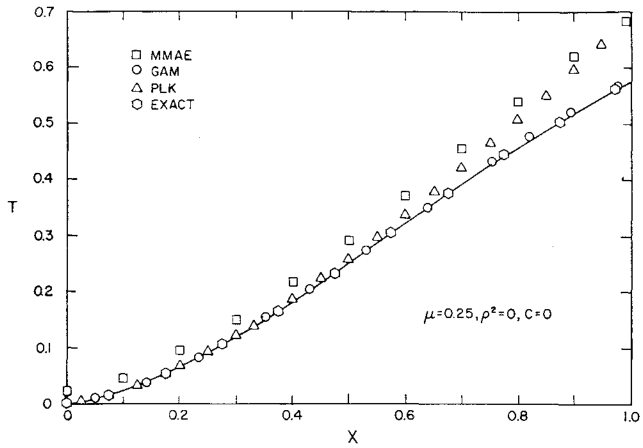

Fig. 4 The time [$T = (2)^{1/2}t$] taken by the spaceship to travel a distance x in the one-dimensional problem when the total energy is zero, and $\mu = 0.25$.

Examining equation (5.3b), we see that $x_1(0) = 0$, and thus $x = 0$ when $s = 0$, i.e., the straining vanishes at $x = 0$. Using the initial conditions, we find that the solution is

$$2^{1/2}t = (1/\rho^3)\sin^{-1}\rho s^{1/2} - (1/\rho^2)[s(1-\rho^2 s)]^{1/2} \quad (5.4a)$$

$$y = -\mu cs \quad (5.4b)$$

where

$$x = s + \mu\left[\frac{2}{\rho^3}\left(\frac{1-\rho^2 s}{s}\right)^{1/2}\sin^{-1}\rho s^{1/2} - \frac{2-\rho^2}{\rho^2(1-\rho^2)} + \frac{1}{2(1-\rho^2)^{3/2}}\left(\frac{1-\rho^2 s}{s}\right)^{1/2}\ln\frac{(1-\rho^2 s)^{1/2} + [(1-\rho^2)s]^{1/2}}{(1-\rho^2 s)^{1/2} - [(1-\rho^2)s]^{1/2}}\right] \quad (5.4c)$$

5.2 Generalized Method

Let

$$t = \sum_{n=0}^{\infty}\mu^n t_n(x,\eta) \quad (5.5a)$$

$$y = \sum_{n=1}^{\infty}\mu^n y_n(x,\eta) \quad (5.5b)$$

where

$$\eta = 1 - x/\mu \quad (5.5c)$$

By interchanging the roles of x and t in (2.2) and by replacing $(1 - x)$ with $\mu\eta$ and leaving the other x's as they are, we transform the ordinary differential equations (2.2) into the following partial differential equations:

$$\frac{t_{xx} - (2/\mu)t_{x\eta} + (1/\mu^2)t_{\eta\eta}}{[t_x - (1/\mu)t_\eta]^3} = \frac{(1-\mu)x}{(x^2+y^2)^{3/2}} - \frac{\mu^2\eta}{(\mu^2\eta^2+y^2)^{3/2}} \quad (5.6a)$$

$$\frac{y_{xx} - (2/\mu)y_{x\eta} + (1/\mu^2)y_{\eta\eta}}{[t_x - (1/\mu)t_\eta]^2} - \left(y_x - \frac{1}{\mu}y_\eta\right)\left[\frac{(1-\mu)x}{(x^2+y^2)^{3/2}} - \frac{\mu^2\eta}{(\mu^2\eta^2+y^2)^{3/2}}\right] = -\frac{(1-\mu)y}{(x^2+y^2)^{3/2}} - \frac{\mu y}{(\mu^2\eta^2+y^2)^{3/2}} \quad (5.6b)$$

Substituting the series (5.5a) and (5.5b) into (5.6) and equating coefficients of equal powers of μ gives

$$t_{0\eta} = 0 \quad (5.7a)$$

† The right-hand sides of Eqs. (4.11) and (4.12) of Lagerstrom and Kevorkian are missing the terms $-(\frac{1}{2})\mu$ and $(\frac{1}{2})\mu$, respectively, and these typographical errors have been rectified.

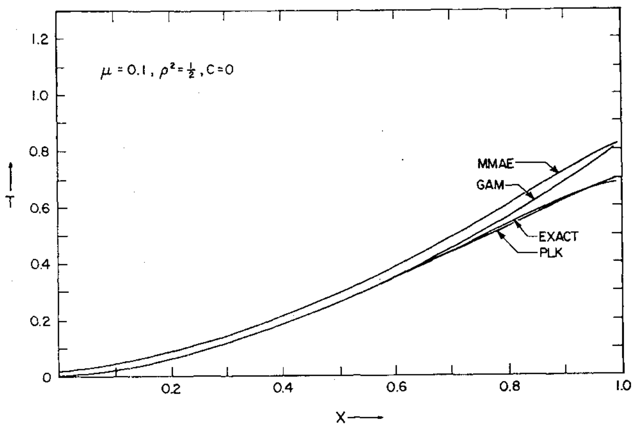

Fig. 5 The time $[T = (2)^{1/2}t]$ taken by the spaceship to travel a distance x in the one-dimensional problem when the total energy is $-\frac{1}{2}$, and $\mu = 0.1$.

$$[t_{1\eta\eta}/(t_{1\eta} - t_{0x})^3] - [\eta/(\eta^2 + y_1^2)^{3/2}] = 0 \quad (5.7b)$$

$$\frac{y_{1\eta\eta}}{(t_{1\eta} - t_{0x})^2} - \frac{t_{1\eta\eta}y_{1\eta}}{(t_{1\eta} - t_{0x})^3} + \frac{y_1 t_{1\eta\eta}}{\eta(t_{1\eta} - t_{0x})^3} = 0 \quad (5.7c)$$

The solution of (5.7a) is

$$2^{1/2}\, t_0 = A(x) \quad (5.8a)$$

where $A(x)$ is an arbitrary function of x. The straightforward perturbation expansion of (5.8a) must give (3.3a). Thus,

$$A(x) = +(1/\rho^3)\sin^{-1}\rho x^{1/2} - (1/\rho^2)[x(1 - \rho^2 x)]^{1/2} \quad (5.8b)$$

Manipulation of Eqs. (5.7) gives

$$\frac{1 + y_{1\eta}^2}{2(t_{1\eta} - t_{0x})^2} - \frac{1}{(\eta^2 + y_1^2)^{1/2}} = B(x) \quad (5.9a)$$

$$\eta y_{1\eta} - y_1/(t_{1\eta} - t_{0x}) = C(x) \quad (5.9b)$$

where $B(x)$ and $C(x)$ are arbitrary functions of x. The solutions of (5.9) are

$$y_1 = \frac{2^{1/2}B^{1/2}C}{1 - 2BC^2}\left[\eta + C^2 \pm \eta\left(1 + \frac{1}{\eta B} + \frac{C^2}{2B\eta^2}\right)^{1/2}\right] \quad (5.10a)$$

$$2^{1/2}t_1 = \frac{1}{B^{1/2}(2BC^2 - 1)}\left[-2\left(\eta + \frac{1}{2B}\right)BC^2 \pm \eta\left(1 + \frac{1}{\eta B} + \frac{C^2}{2\eta^2 B}\right)^{1/2}\right] - \frac{1}{2B^{3/2}} \times \sinh^{-1}\frac{2B}{(2BC^2 - 1)(1 + 2BC^2)^{1/2}}\left[-2BC^2\left(\eta + \frac{1}{2B}\right) \pm \eta\left(1 + \frac{1}{\eta B} + \frac{C^2}{2\eta^2 B}\right)^{1/2}\right] + 2^{1/2}t_{0x}\eta + D(x) \quad (5.10b)$$

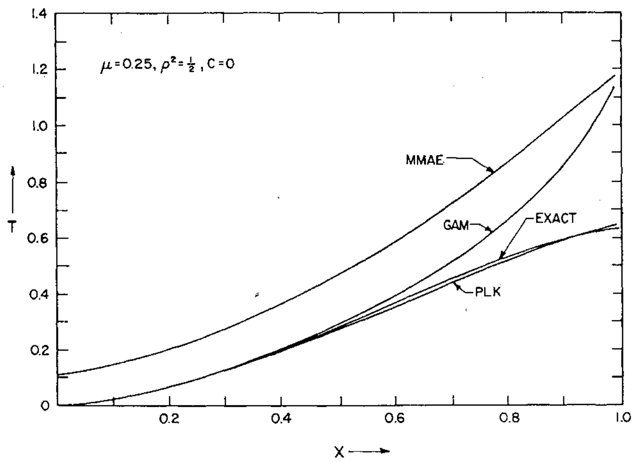

Fig. 6 The time $[T = (2)^{1/2}t]$ taken by the spaceship to travel a distance x in the one-dimensional problem when the total energy is $-\frac{1}{2}$, and $\mu = 0.25$.

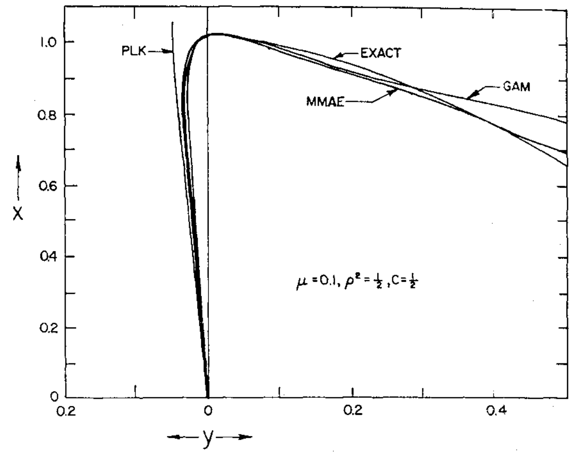

Fig. 7 Trajectory of the spaceship for the case $c = \frac{1}{2}$, and $\mu = 0.1$.

where $D(x)$ is an arbitrary function of x. The condition that $[t_1(x, \eta)]/[t_0(x, \eta)]$ is bounded for $0 \le \eta \le (a/\mu)$, $a = \max x$ on trajectory, requires that

$$2^{1/2}t_{0x} - (1/B^{1/2}) = 0 \quad (5.11a)$$

or

$$B(x) = 1 - \rho^2 x/x \quad (5.11b)$$

The straightforward expansions of (5.10a) and (5.10b) must give (3.3c) and (3.3b), respectively. Hence,

$$C(x) = c[2x(1 - \rho^2 x)]^{1/2} \quad (5.12a)$$

$$D(x) = -\frac{2}{\rho^3}\sin^{-1}\rho x^{1/2} + \frac{2 - \rho^2}{\rho^2(1 - \rho^2)}\left(\frac{x}{1 - \rho^2 x}\right)^{1/2} + \frac{1}{2}\left(\frac{x}{1 - \rho^2 x}\right)^{3/2} - \frac{1}{2}\left(\frac{x}{1 - \rho^2 x}\right)^{3/2} \times \left\{\ln\frac{1 - x}{\mu} + 2\ln 2 + \ln\frac{1 - \rho^2 x}{x[1 + 4c^2(1 - \rho^2 x)^2]^{1/2}}\right\} + \frac{1}{2(1 - \rho^2)^{3/2}} \times \ln\frac{1 + (1 - 2\rho^2)x - 2[(1 - \rho^2)x(1 - \rho^2 x)]^{1/2}}{1 - x} \quad (5.12b)$$

Therefore,

$$y = \frac{2c(1 - \rho^2 x)}{1 - 4c^2(1 - \rho^2 x)^2}\left\{\eta + 2c^2 x(1 - \rho^2 x) \pm \eta\left[1 + \frac{x}{\eta(1 - \rho^2 x)} + \frac{c^2 x^2}{\eta^2}\right]^{1/2}\right\}\mu \quad (5.13a)$$

$$2^{1/2}t = \frac{1}{\rho^3}\sin^{-1}\rho x^{1/2} - \frac{1}{\rho^2}[x(1 - \rho^2 x)]^{1/2} + \mu\left\{\eta\left(\frac{x}{1 - \rho^2 x}\right)^{1/2} + D(x) + \left(\frac{x}{1 - \rho^2 x}\right)^{1/2} \times \frac{E(x, \eta)}{4c^2(1 - \rho^2 x)^2 - 1} - \frac{1}{2}\left(\frac{x}{1 - \rho^2 x}\right)^{3/2} \times \sinh^{-1}\frac{2(1 - \rho^2 x)E(x, \eta)}{x[4c^2(1 - \rho^2 x)^2 - 1][1 + 4c^2(1 - \rho^2 x)^2]^{1/2}}\right\} \quad (5.13b)$$

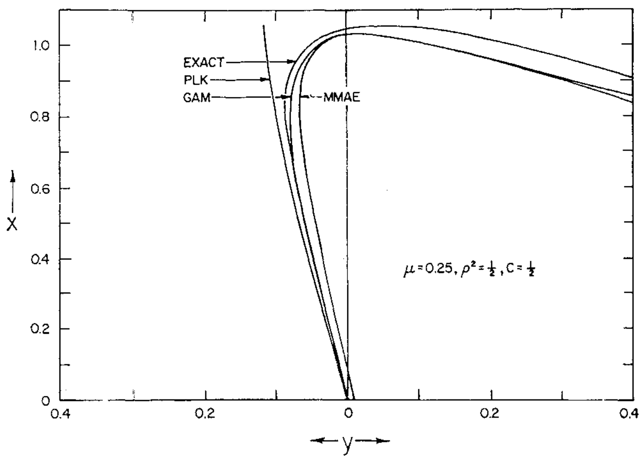

Fig. 8 Trajectory of the spaceship for the case $c = \frac{1}{2}$, and $\mu = 0.25$.

where

$$E(x, \eta) = -4c^2(1 - \rho^2 x)\eta - 2c^2 x(1 - \rho^2 x) \mp \eta\{1 + [x/(1 - \rho^2 x)\eta] + (c^2x^2/\eta^2)\}^{1/2} \quad (5.13c)$$

Here the positive and negative signs are taken for the motions toward and away from the moon, respectively. The results of this section are shown in Figs. 5–8 for comparison with the exact solution[10] and the solution of Lagerstrom and Kevorkian[2] for $c = 0$, $\rho^2 = \frac{1}{2}$ and $c = \frac{1}{2}$, $\rho^2 = \frac{1}{2}$, respectively.

6. Conclusion

The PLK method gives divergent trajectories, as shown in Figs. 7 and 8, although it gives excellent results for the one-dimensional case. The reason is that the one-dimensional problem has a singularity at $x = 1 + [\mu/(1 - \rho^2)] + 0(\mu^2)$, which is outside the domain of interest $0 \leq x \leq 1$. The straining of x moves the singularity from $x = 1$ toward its right position $x = 1 + [\mu/(1 - \rho^2)] + 0(\mu^2)$. On the other hand, the singularity in the two-dimensional case is not, in general, within a distance of $0(\mu)$ from the artificial singularity $x = 1$ of the straightforward perturbation expansion. Furthermore, there is a sharp change in the direction of the angular momentum after moon passage, as seen from the exact solution in Figs. 7 and 8. Thus, the straining, which is of $0(\mu)$, cannot push the singularity from $x = 1$ to its right position and cannot produce the sharp change in the angular momentum direction.

The generalized asymptotic method (GAM) gives closer approximations to the exact solutions than the method of matched asymptotic expansions (MMAE), as seen in Figs. 2–8. The reason is that, in minimizing the error, the intermediate region is not treated equally with the outer and inner regions if we use the method of matched asymptotic expansions. On the other hand, the whole domain is treated equally well when we use the generalized asymptotic method. Thus, the generalized method can be used for wider range of the small parameter μ. The problem considered here has two other nonuniformities, one when t is large and the other when $\rho = 1$.

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